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Symmetrized tensor products of induced representations

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Abstract. A solution to the general problem of decomposing any tensor power of an induced representation symmetrized in accordance with a given representation of the appropriate symmetric group was given in 1973 by Gard. In this paper a shorter and more general proof of the same results is provided. The method used has other applications (albeit not very exciting ones) in quite different areas of representation theory.

1. Notation and prerequisites

Our notation will generally follow that of Mackey (1949, 1968), but for the reader who may be familiar with a more explicit version of the inducing construction, we summarize its salient features.

Suppose that the closed subgroup K of a separable locally compact group G has a unitary representation D in a Hilbert space $\mathfrak{h}(D)$. This can be induced up to a unitary representation U of G, which acts on functions ψ from G to $\mathfrak{h}(D)$ satisfying

$$\psi(kg) = D(k)\psi(g) \qquad \text{for } k \text{ in } K \text{ and } g \text{ in } G,$$

and also

$$\int_{K\backslash G} \|\psi(g)\|^2 < \infty$$

where the integration is with respect to the unique quasi-invariant measure class on the coset space $K\backslash G$. The induced representation can now be defined by

$$(U_x\psi)(g) = \sqrt{\rho(g, x)}\psi(gx) \qquad \text{for } x \text{ and } g \text{ in } G,$$

where $\rho(g, x)$ is the Radon–Nikodym derivative needed to compensate for the possible non-invariance of the measure used and make the representation U unitary. (For finite, and more generally compact and semi-simple Lie groups, this factor can be omitted.)

If we now wish to restrict our attention to another subgroup H of G we have $(U_h\psi)(g) = \sqrt{\rho(g, h)}\psi(gh)$ for g in G and h in H. The action of H on the argument of the function, together with the K-equivariance condition, allows us to change g to any other element in the same double coset of $K\backslash G/H$, but will never take us outside it. Let us therefore pick some element α to represent this double coset, and write

$$\psi_\alpha(h) = \psi(\alpha h) \qquad \text{for } h \text{ in } H.$$

Now if k is in both H and $K^\alpha = \alpha^{-1}K\alpha$ we have

$$\psi_\alpha(kh) = \psi(\alpha k\alpha^{-1}\alpha h) = D(\alpha k\alpha^{-1})\psi(\alpha h).$$

Writing $D^\alpha(k)$ for $D(\alpha k\alpha^{-1})$ we obtain

$$\psi_\alpha(kh)=D^\alpha(k)\psi_\alpha(h)$$

and this is all that survives of the K-equivariance condition on functions of this type. In this way the induced representation can be decomposed into a direct integral over α in $K\backslash G/H$ of the representations on the functions ψ_α, and the form of these is that each is a representation of H induced from the representation D^α of the subgroup $H\cap K^\alpha$. This is the content of Mackey's subgroup theorem (1951, 1968). (There is a technical condition that K and H be 'regularly related', but once this is satisfied the decomposition works for projective representations as well as ordinary ones.)

2. Representations of groups with automorphisms

Our derivation of the decomposition formula for a symmetrized tensor product breaks up into three parts. The first of these is to apply the Mackey decomposition of the preceding section to a particular semi-direct product group which we now introduce.

To avoid repetition we shall assume henceforth that all the groups mentioned are separable and locally compact. Suppose that G is such a group acted on by a group of automorphisms Π. Suppose further that K is a closed subgroup of G which is invariant under the action of Π, and that H is another closed subgroup, each of whose elements is fixed by all the automorphisms of Π.

(This is motivated by the following example. Take G to be the direct product of n copies of a group $\mathscr{G}$, K the direct product of n copies of $\mathscr{K}$ ($\mathscr{K}$ a closed subgroup of $\mathscr{G}$), H the diagonal subgroup $\{(g, g, \ldots, g)\in G\colon g\in\mathscr{G}\}$, and Π the group S_n where the permutation π acts by taking $(g_1, \ldots, g_n)$ in G to $[(g_1, \ldots, g_n)]\pi=(g_{\pi(1)}, \ldots, g_{\pi(n)})$. The n-fold tensor product of a representation V of $\mathscr{G}$ can be regarded as the restriction to the diagonal subgroup of the outer product representation, $(g_1, \ldots, g_n)\mapsto V(g_1)\otimes\ldots\otimes V(g_n)$ of G. If V is induced from $\mathscr{K}$ then the outer product representation is induced from K. Symmetrization of the tensor product can now be effected by looking at the action of Π. We would like to thank the referee for bringing to our attention the paper of Kerber (1973) in which such wreath products are also used.)

Returning to the general situation, we write $[g]\pi$ for the result of acting on the group element g in G with the automorphism π. We can then form the semi-direct product, $\Pi \circledS G$, whose multiplication law is

$$(\pi_1g_1)(\pi_2g_2)=(\pi_1\pi_2)([g_1]\pi_2g_2) \qquad \text{for } \pi_i \text{ in } \Pi,\ g_i \text{ in } G,\ i=1,2.$$

Within this semi-direct product $[g]\pi=\pi^{-1}g\pi$.

The semi-direct product of K with Π can similarly be formed. In the case of H, the trivial action of Π on this subgroup gives simply the direct product $\Pi\times H$.

Let us now suppose that D is a representation of K which is taken into an equivalent representation under the action of Π. (That is, if we define $D^\pi(k)=D([k]\pi^{-1})$, then D^π is equivalent to D for each π in Π.) According to the theory of representations of group extensions developed by Mackey (1958), we can extend D to a projective representation Δ of $\Pi\circledS K$, which when restricted to K gives D. From Δ a representation of $\Pi\circledS G$ may be induced and then restricted down to $\Pi\times H$. (If we wish to work with projective representations of G we may do so, provided that the multiplier lifts to $\Pi\circledS G$.)

The subgroup theorem of Mackey now tells us how to reduce this to a direct integral over the double coset space $\Pi \circledS K \backslash \Pi \circledS G / \Pi \times H$ of representations induced from subgroups of the form $(\Pi \times H) \cap (\Pi \circledS K)^\alpha = L^\alpha$, say, where α is a double coset representative. Now the space of double cosets can be identified with the space $K \backslash G/H$ factored out by the natural action of Π: $(KgH)\pi = K([g]\pi)H$. (This is well-defined by virtue of the invariance of K and H under Π.) In the special case of the tensor products it is this further action of Π which effects the symmetrization. It collects many $K : H$ double cosets together into a single point of $(\Pi \circledS K) \backslash (\Pi \circledS G)/(\Pi \times H)$. One way to see how this identification comes about is to notice that $K \backslash G$ can be identified naturally with $\Pi \circledS K \backslash \Pi \circledS G$ on which there is still the action of $\Pi \times H$ to be taken into account.

One useful observation which comes out of this identification is that the double coset representative α can always be chosen to lie in G and not just in $\Pi \circledS G$.

3. The general theorem

The observations of § 2 lead us to investigate the representation of $\Pi \times H$ induced from the representation Δ^α of L^α. First we shall introduce some of the subgroups which play a part in the theory.

We may define $\Pi_\alpha = \Pi \cap L^\alpha$, and $H_\alpha = H \cap L^\alpha$. These are the stabilizers in Π and H respectively of the coset L^α in $L^\alpha \backslash \Pi \times H$. We also need

$$\Pi(\alpha) = \{\pi \in \Pi : (\pi h) \in L^\alpha \text{ for some } h \text{ in } H\}$$

and

$$H(\alpha) = \{h \in H : (\pi h) \in L^\alpha \text{ for some } \pi \text{ in } \Pi\}.$$

Clearly Π_α and H_α are subgroups of $\Pi(\alpha)$ and $H(\alpha)$ respectively, but we can, in fact, say more than this. The projection from L^α to $\Pi(\alpha)$ defined by $(\pi, h) \mapsto \pi$ is clearly a homomorphism, and its kernel is H_α, so that $H_\alpha \backslash L^\alpha$ is isomorphic to $\Pi(\alpha)$. Similarly $\Pi_\alpha \backslash L^\alpha$ is isomorphic to $H(\alpha)$. On factoring out further we deduce that

$$\Pi_\alpha \backslash \Pi(\alpha) \cong \Pi_\alpha \times H_\alpha \backslash L^\alpha \cong H_\alpha \backslash H(\alpha).$$

We can easily write down the correspondences explicitly, for if π is in $\Pi(\alpha)$ and $h(\pi)$ in $H(\alpha)$ is chosen so that $\pi h(\pi)$ is in L^α, then

$$\Pi_\alpha \pi \to \Pi_\alpha \times H_\alpha(\pi h(\pi)) \to H_\alpha h(\pi).$$

The element $h(\pi)$ may clearly be chosen to depend only on the coset in which π lies. Because the maps are onto the respective quotient groups we deduce that the most general element of $H(\alpha)$ must have the form $kh(\pi)$ for some k in H_α and some π in $\Pi(\alpha)$. Similarly the most general element of L^α has the form $\pi k h(\pi)$.

The induced representation in which we are interested takes place on functions ψ from $\Pi \times H$ to $\mathfrak{h}(D)$ which satisfy

$$\psi(lg) = \Delta^\alpha(l)\psi(g) \qquad \text{for } l \text{ in } L^\alpha,\ g \text{ in } \Pi \times H.$$

We now want to pick out of this representation a subrepresentation which transforms like a given representation W of Π. When Π is compact, as it is in our motivating example, the projection which picks out the relevant subspace can be given explicitly as

$$\int_\Pi \overline{\chi_W(\pi)}\, U_\pi \, d\pi$$

where χ_W is the character of W, and $d\pi$ is the normalized Haar measure on Π. We shall therefore assume that we can find the subspace of functions on which U acts in a way equivalent to W. To be explicit, let S be the unitary operator which provides the equivalence, so that for ψ in the subspace

$$U_\pi \psi = S^{-1} W_\pi S \psi.$$

There is, of course, a consistency condition to be satisfied in order that this be possible. This is that if π is in Π_α and g is in H then

$$(S^{-1} W_\pi S\psi)(g) = (U_\pi \psi)(g) = \psi(g\pi) = \psi(\pi g) = \Delta^\alpha(\pi)\psi(g).$$

This can be regarded as saying that W and Δ^α each restricted to Π_α are not disjoint.

All but the last of the above chain of equalities is true for any π in Π, so that once we know the values of the function on elements of H the rest can be deduced using

$$\psi(\pi g) = (S^{-1} W_\pi S\psi)(g).$$

We can now look to see what remains of the equivariance condition on functions. Any element of L^α can be written in the form $l(\pi) = \pi k h(\pi)$ for some π in $\Pi(\alpha)$ and k in H_α. Then

$$\psi(l(\pi)g) = \Delta^\alpha(l(\pi))\psi(g).$$

Since π commutes with elements of H the left-hand side can also be written as $\psi(kh(\pi)g\pi) = (S^{-1} W_\pi S\psi)(kh(\pi)g)$, so that

$$(S^{-1} W_\pi S\psi)(kh(\pi)g) = \Delta^\alpha(l(\pi))\psi(g).$$

On replacing $S^{-1} W_\pi S\psi$ by ψ:

$$\psi(kh(\pi)g) = \Delta^\alpha(l(\pi))(S^{-1} W_\pi^{-1} S\psi)(g).$$

This is entirely equivalent to our original equivariance condition on functions in the subspace. It can also be written in another form. Any $l(\pi)$ can be expressed as πh for some h in $H(\alpha)$, and π in $\Pi(\alpha)$. Moreover, for any h in $H(\alpha)$ there is a $\pi(h)$ such that $\pi(h)h$ is in L^α. Thus we may write equivalently the condition that

$$\psi(hg) = \Delta^\alpha(\pi(h)h)(S^{-1} W_{\pi(h)}^{-1} S\psi)(g)$$

for g in H and h in $H(\alpha)$.

In general this is not a terribly easy condition to use because of the way in which it involves the action W of $\Pi(\alpha)$. However, if W is one-dimensional, then the condition simplifies considerably to

$$\psi(hg) = W(\pi(h))^{-1} \Delta^\alpha(\pi(h)h)\psi(g)$$

which is just the condition for the representation to be induced from the representation $h \mapsto W(\pi(h))^{-1} \Delta^\alpha(\pi(h)h)$ of $H(\alpha)$.

The consistency condition is just that

$$\Delta^\alpha(\pi)\psi(g) = W(\pi)\psi(g)$$

for all $\pi \in \Pi_\alpha$, g in H and ψ in the subspace. That is, Δ^α restricted to Π_α contains the restricted character W.

Sometimes it is possible to simplify this requirement by making a careful choice of the double coset representative α, so that α is simply invariant under the action of Π_α. In that case Δ^α and Δ agree when restricted to Π_α, and the consistency requirement is just that Δ restricted to Π is not disjoint from the restriction of W. In particular, if this is to be true at the double coset KH where we can take the representative $\alpha = 1$, it is necessary that Δ contains W.

The results of this section can be summarized in the following theorem.

Let U be the unitary representation of G induced from the representation D of K. That part of the representation space which transforms under the representation W of Π is invariant under the restriction of U to H, and, as a representation space for H, it can be written as a direct integral over those α in $(K\backslash G/H)/\Pi$ for which Δ^α and W restricted to Π_α are not disjoint, of representations on functions $\psi_\alpha: H \mapsto \mathfrak{h}(D)$ which satisfy

$$\psi_\alpha(hg) = \Delta^\alpha(\pi(h)h)(S^{-1}W^{-1}_{\pi(h)}S\psi)(g)$$

for all g in H and h in $H(\alpha)$.

Equivalently the ψ_α satisfy

$$\psi_\alpha(kh(\pi)g) = \Delta^\alpha(\pi kh(\pi))(S^{-1}W_\pi^{-1}S\psi_\alpha)(g)$$

for all k in H_α, g in H, and at least one π in each coset of $\Pi_\alpha \backslash \Pi(\alpha)$.

If W is one-dimensional, then the contribution of α to the direct integral is the representation of H induced from the representation $h \mapsto W(\pi(h))^{-1}\Delta^\alpha(\pi(h)h)$ of $H(\alpha)$.

4. Symmetrized tensor powers

The final step is to apply this result to the example mentioned in § 2. We therefore take

$$G = \mathscr{G} \times \ldots \times \mathscr{G}\ (n \text{ copies})$$

$$K = \mathscr{K} \times \ldots \times \mathscr{K}\ (n \text{ copies})$$

$$H = \{(g, \ldots, g) \in G : g \in \mathscr{G}\}$$

(for brevity we shall write $\tilde{g}$ instead of $(g, \ldots, g)$), and

$$\Pi = S_n$$

acting as permutations of the components.

We may pick double coset representatives $\alpha = (\alpha_1, \ldots, \alpha_n)$ in such a way that $\alpha_1 = 1$, α_2 represents a double coset in $\mathscr{K}\backslash\mathscr{G}/\mathscr{K}$, α_3 represents a double coset in $\mathscr{K}\backslash\mathscr{G}/\mathscr{K} \cap \mathscr{K}^{\alpha_2}$, and so on till α_n represents a double coset in $\mathscr{K}\backslash\mathscr{G}/\mathscr{K} \cap \mathscr{K}^{\alpha_2} \cap \ldots \cap \mathscr{K}^{\alpha_{n-1}}$. As far as possible we duplicate representatives, that is, for $j \geq i$, we use α_i to represent the $\mathscr{K}$: $\mathscr{K} \cap \mathscr{K}^{\alpha_2} \cap \ldots \cap \mathscr{K}^{\alpha_j}$ double coset in which it lies. One way of achieving this is to write α_i in the form of a product $\alpha_i^1\alpha_i^2 \ldots \alpha_i^i$, where α_i^1 is the representative of the double coset $\mathscr{K}\alpha_i\mathscr{K}$ chosen from some standard set, $\alpha_i^1\alpha_i^2$ is a standard representative for the double coset $\mathscr{K}\alpha_i(\mathscr{K} \cap \mathscr{K}^{\alpha_2})$, likewise, and so on. Now, having included a representative α we need not include any representative for its permutations $K[\alpha]\pi H$. That is, we pick one representative in each orbit of Π.

Now $\alpha\pi\tilde{h} = \pi[\alpha]\pi\tilde{h} = \pi(\alpha_{\pi(1)}h, \ldots, \alpha_{\pi(n)}h)$, and $\pi\tilde{h}$ lies in L^α provided that this lies in $(\Pi \circledS K)\alpha$. This forces $\alpha_{\pi(j)}h$ to lie in $\mathscr{K}\alpha_j$ for each $j = 1, \ldots, n$, so h has to lie in

$$\bigcap_{j=1}^{n} \alpha^{-1}_{\pi(j)}\mathscr{K}\alpha_j$$

and π has to be such that this is not empty. From this we deduce that

$$\Pi_\alpha=\{\pi\in\Pi:\ \alpha_{\pi(j)}\in\mathcal{K}\alpha_j,\ j=1,\dots,n\}$$

$$H_\alpha=\left(\bigcap_{j=1}^{n}\alpha_j^{-1}\mathcal{K}\alpha_j\right)^{\sim}=(\mathcal{K}\cap\mathcal{K}^{\alpha_2}\cap\dots\cap\mathcal{K}^{\alpha_{n-1}})^{\sim}$$

$$\Pi(\alpha)=\left\{\pi\in\Pi:\ \bigcap_{j=1}^{n}\alpha_{\pi(j)}^{-1}\mathcal{K}\alpha_j\neq\varnothing\right\}$$

$$H(\alpha)=\left(\bigcup_{\pi\in\Pi}\left(\bigcap_{j=1}^{n}\alpha_{\pi(j)}^{-1}\mathcal{K}\alpha_j\right)\right)^{\sim}$$

The expression for Π_α can certainly be simplified, for if $\alpha_{\pi(j)}$ is in $\mathcal{K}\alpha_j$ then $\mathcal{K}\alpha_j$ and $\mathcal{K}\alpha_{\pi(j)}$ are equal, and by our convention this can only happen if $\alpha_j=\alpha_{\pi(j)}$ since we choose the same representatives wherever possible. Accordingly we deduce that

$$\Pi_\alpha=\{\pi\in\Pi:\ \alpha_j=\alpha_{\pi(j)},\ j=1,\dots,n\}.$$

The elements of Π_α just serve to permute the equal components of α. Let us suppose that the distinct components of α are $\beta_1,\dots,\beta_t$, and that β_j occurs in n_j places, $(n_1+\dots+n_t=n)$. Then Π_α is isomorphic to $S_{n_1}\times S_{n_2}\times\dots\times S_{n_t}$.

The same convention also helps us to simplify the form of $\Pi(\alpha)$. Suppose that $\alpha_j=\alpha_k$. We know that, for π in $\Pi(\alpha)$, $\alpha_{\pi(j)}^{-1}\mathcal{K}\alpha_j$ and $\alpha_{\pi(k)}^{-1}\mathcal{K}\alpha_k$ have an element in common, and so $\mathcal{K}\alpha_{\pi(j)}$ and $\mathcal{K}\alpha_{\pi(k)}$ have a non-empty intersection. They must therefore coincide, and, by our convention, this forces $\alpha_{\pi(j)}=\alpha_{\pi(k)}$. Thus an element of $\Pi(\alpha)$ can only move equal components of α to equal components. If we write $\pi(\beta_i)$ for the component into which π permutes β_i then π is in $\Pi(\alpha)$ provided that $\bigcap_{j=1}^{t}\beta_j^{-1}\mathcal{K}\pi(\beta_j)$ is not empty.

As far as the representation theory is concerned we shall take for D a representation of the form $A\otimes\dots\otimes A$, where A is a representation of $\mathcal{K}$. This extends to ΠⓈK when we put

$$\Delta(\pi(k_1,\dots,k_n))v_1\otimes\dots\otimes v_n=A(k_{\pi^{-1}(1)})v_{\pi^{-1}(1)}\otimes\dots\otimes A(k_{\pi^{-1}(n)})v_{\pi^{-1}(n)}.$$

If we introduce the notation E_π for the operator on $\mathfrak{h}(A)\otimes\dots\otimes\mathfrak{h}(A)$ which permutes the factors as π, then we may write this as

$$\Delta(\pi(k_1,\dots,k_n))=A(k_{\pi^{-1}(1)})\dots A(k_{\pi^{-1}(n)})E_\pi.$$

If $\pi\tilde{h}$ is in L^α then $\Delta^\alpha(\pi\tilde{h})$ can easily be found to be

$$\Delta(\pi(\alpha_{\pi(1)}h\alpha_1^{-1},\dots,\alpha_{\pi(n)}h\alpha_n^{-1}))=(A(\alpha_1h\alpha_{\pi^{-1}(1)}^{-1})\otimes\dots\otimes A(\alpha_nh\alpha_{\pi^{-1}(n)}^{-1}))E_\pi.$$

We can now deduce the consistency requirement by putting $h=1$, and taking π in Π_α. Then $\Delta^\alpha(\pi)=E_\pi$, and we require that E restricted to Π_α should not be disjoint from W. This condition is discussed in the final section of Gard (1973) and we have nothing to add to what is said there. For simplicity we shall from now on consider only the cases of the symmetrized and antisymmetrized tensor products for which W is one-dimensional. In that case the consistency condition is that E restricted to Π_α contains the symmetric or alternating representation as appropriate, and this always happens.

The induced representation is now defined on functions from H to $\mathfrak{h}(A)\otimes\dots\otimes\mathfrak{h}(A)$ which satisfy

$$\psi(\tilde{h}g)=\Delta^\alpha(\tilde{h}\pi(h))W(\pi(h))^{-1}\psi(g)$$

for in $\bigcup_{\pi\in\Pi}(\bigcap_{j=1}^{n}\alpha_{\pi(j)}^{-1}\mathcal{H}\alpha_j)$, g in $\mathcal{G}$, and $\pi=\pi(h)$ a permutation such that $\alpha_{\pi(j)}^{-1}\mathcal{H}\alpha_j$ contains h.

We may as well drop the tilde on the argument of ψ and consider the functions as defined on $\mathcal{G}$. Using the formula for Δ^{α} on L this leads to the equivariance condition

$$\psi(hg)=\overline{W(\pi)}A(\alpha_1 h\alpha_{\pi^{-1}(1)}^{-1})\otimes\ldots\otimes A(\alpha_n h\alpha_{\pi^{-1}(n)}^{-1})E_\pi\psi(g)$$

where we have written π for $\pi(h)$. An equivalent form is that

$$\psi_\alpha(kk(\pi)g)=\overline{W(\pi)}A(\alpha_1 kk(\pi)\alpha_{\pi^{-1}(1)}^{-1})\otimes\ldots\otimes A(\alpha_n kk(\pi)\alpha_{\pi^{-1}(n)}^{-1})E_\pi\psi_\alpha(g)$$

for g in $\mathcal{G}$, k in $\mathcal{H}\cap\mathcal{H}^{\alpha_2}\cap\ldots\cap\mathcal{H}^{\alpha_n}$ and $k(\pi)$ is an element in $\bigcap_{j=1}^{n}\alpha_{\pi(j)}^{-1}\mathcal{H}\alpha_j$. ($\widetilde{k(\pi)}$ is just the element $h(\pi)$ introduced earlier).

As an example, we consider in detail the case of the symmetrized or antisymmetrized square of a finite group representation where our results can be compared directly with those of Mackey (1953). For this we take $n=2$. Since $\alpha_1=1$ we need only specify $\alpha_2=\beta$ which is a double coset representative for $\mathcal{H}\backslash\mathcal{G}/\mathcal{H}$. The transposition τ in S_2 takes $(1,\beta)$ to $(\beta,1)=(1,\beta^{-1})(\beta,\beta)$, so that we need only represent one of $\mathcal{H}\beta\mathcal{H}$ and $\mathcal{H}\beta^{-1}\mathcal{H}$, unless, of course, they coincide. If they are the same then, in Mackey's terminology we say that $\mathcal{H}\beta\mathcal{H}$ is self-inverse.

Now by our earlier discussion the elements of $\Pi(\alpha)$ are just those permutations which take a double coset into another having a represenative with the same blocks of equal components though possibly permuted amongst themselves. Since τ takes $\mathcal{H}\beta\mathcal{H}$ to $\mathcal{H}\beta^{-1}\mathcal{H}$, τ is in $\Pi(\alpha)$ just when $\mathcal{H}\beta\mathcal{H}$ is self-inverse (so that $\mathcal{H}\beta^{-1}\mathcal{H}$ also has representative β).

If $\mathcal{H}\beta\mathcal{H}$ is not self-inverse, therefore, $\Pi(\alpha)=\{1\}$, and we have (writing ψ_β instead of $\psi_{(1,\beta)}$) just the condition

$$\psi_\beta(kg)=A(k)\otimes A(\beta k\beta^{-1})\psi_\beta(g)$$

for all g in $\mathcal{G}$ and all k in $\mathcal{H}\cap\beta^{-1}\mathcal{H}\beta$.

If $\mathcal{H}\beta\mathcal{H}$ is self-inverse then $\Pi(\alpha)=S_2$ and in addition to the above condition we also have

$$\psi_\beta(kk(\pi)g)=\pm A(kk(\pi)\beta^{-1})\otimes A(\beta kk(\pi))E_\pi\psi_\beta(g)$$

where the sign is chosen according to whether we wish to symmetrize or antisymmetrize the representation.

The case of $\beta=1$ is somewhat special as then $\Pi_\alpha=S_2$ also, and by virtue of the consistency condition which requires ψ_1 to be appropriately symmetrized the second condition reduces to the first, each giving

$$\psi_1(kg)=A(k)\otimes A(k)\psi_1(g)$$

for k in $\mathcal{H}$.

Thus the symmetric or antisymmetric square decomposes into a direct sum of three parts, one being for $\beta=1$, one a sum over self-inverse double cosets, and one a sum over $\mathcal{H}\beta\mathcal{H}$ which are not self-inverse (picking just one out of $\mathcal{H}\beta\mathcal{H}$ and $\mathcal{H}\beta^{-1}\mathcal{H}$). Apart from the fact that we have not identified the tensor product with an operator algebra this is precisely Mackey's conclusion.

We can in general summarize the effect of symmetrization as follows. First it enables us to reduce the number of double cosets needed in the Mackey decomposition. Second it enables us to induce from $\bigcup_{\pi\in\Pi}(\bigcap_{j=1}^{n}\alpha_{\pi(j)}^{-1}\mathscr{K}\alpha_j)$ instead of from its subgroup $\mathscr{K}\cap\mathscr{K}^{\alpha_2}\cap\ldots\cap\mathscr{K}^{\alpha_{n-1}}$, which is what we should use in order to obtain an unsymmetrized product. For continuous groups the latter effect tends to be less important than the former, inasmuch as the sort of coincidences of components of α which lead to larger subgroups generally arise in sets of measure zero in the decomposition.

To conclude this section we remark that if the representation A is one-dimensional then E_π is the identity operator and the equivariance condition reduces to

$$\psi(hg)=\overline{W(\pi)}\prod_{j=1}^{n}A(\alpha_j h\alpha_{\pi^{-1}(j)}^{-1})\psi(g).$$

5. The application to semi-direct products

If $\mathscr{G}$ has the form of a semi-direct product of an abelian normal subgroup N with a group of its automorphisms M, then the general theory can be taken a little further. (This case includes, of course, the examples of the Poincaré group, the Euclidean groups, and symmorphic space groups.) The simplifications which occur in this case have also been investigated by Backhouse and Gard (1974).

We shall suppose than an irreducible representation of $\mathscr{G}=NⓢM$ has been formed following Mackey's procedure (1949, 1968), by taking a character ν of N and a representation B of the little group M_ν, and inducing $A=\nu B$. Our group $\mathscr{K}$ is therefore $NⓢM_\nu$.

Since $\mathscr{K}\backslash\mathscr{G}=NⓢM_\nu\backslash NⓢM=M_\nu\backslash M$, all double coset representatives may be chosen to lie in M. Then $\mathscr{K}^{\alpha_i}=NⓢM_\nu^{\alpha_i}$, and $\alpha_1=1$, α_2 represents a double coset in $M_\nu\backslash M/M_\nu$; α_3 a double coset in $M_\nu\backslash M/M_\nu\cap M_\nu^{\alpha_2}$ and so on.

The subgroup from which we induce the appropriately symmetrized tensor product is then

$$\bigcup_{\pi\in\Pi}\left(\bigcap_{j=1}^{n}\alpha_{\pi(j)}^{-1}(NⓢM_\nu)\alpha_j\right)=Nⓢ\left(\bigcup_{\pi\in\Pi}\left(\bigcap_{j=1}^{n}\alpha_{\pi(j)}^{-1}M_\nu\alpha_j\right)\right)=NⓢM'_\nu,\ \text{say.}$$

The equivariance condition says that

$$\psi(x\xi y\eta)=\overline{W(\pi)}\left(\prod_{j=1}^{n}\nu(\alpha_j x\alpha_j^{-1})\right)B(\alpha_1\xi\alpha_{\pi^{-1}(1)}^{-1})\otimes\ldots\otimes B(\alpha_n\xi\alpha_{\pi^{-1}(n)}^{-1})E_\pi\psi(y\eta)$$

for x, y in N, ξ in M'_ν and η in M. Using this our functions can be replaced by functions defined only on M. As in the previous seçtion this collapses down dramatically if the inducing representation is one-dimensional. Then

$$\psi(x\xi y\eta)=\overline{W(\pi)}\prod_{i=1}^{n}\nu(\alpha_i x\alpha_i^{-1})\prod_{j=1}^{n}B(\alpha_j\xi\alpha_{\pi^{-1}(j)}^{-1})\psi(y\eta).$$

This formula applies in the case of irreducible representations of the three-dimensional Euclidean group, which are induced from $\boldsymbol{R}^3ⓢSO(2)$, and also for many of the representations of symmorphic space groups.

Returning to the general case, if we wish to decompose a symmetrized tensor product into irreducibles we may induce in stages, first up to $NⓢM_\nu$ and then on. The decomposition will depend just on how the intermediate representation of $NⓢM_\nu$ decomposes into irreducibles (cf Mackey 1970).

6. Other examples

There are, of course, many examples of groups with automorphisms which fit into the general theory developed in §§ 2 and 3 without looking even remotely like the tensor products treated subsequently. Unfortunately these usually turn out either to be devoid of physical interest or simpler to tackle directly. Nonetheless we shall mention a few to give some idea of the scope of the theory.

One interesting class of examples comes from taking for Π the inner automorphisms by a subgroup P. Let us write π_p for conjugation by the element p in P, that is $[g]\pi_p = p^{-1}gp$. In order that the general theory apply we require that P should normalize K and centralize H. It is also necessary that the representation D of K should extend to a representation Δ of the subgroup generated by K and P. If P happens to be contained in K then this is, of course, automatic. Π can be characterized as the set of π_p such that the commutator $\alpha p\alpha^{-1}p^{-1}$ lies in K, and $\Pi(\alpha)$ is the set of π_p such that $\alpha pH\alpha^{-1}p^{-1}$ intersects K. (If $P \subseteq K$ then these reduce to the requirement that $\alpha p\alpha^{-1}$ be in K and that $\alpha pH\alpha^{-1}p^{-1}$ intersect K respectively. Π_α is thus the group of conjugations by elements of $P_\alpha = P \cap K^\alpha$.) Specifying a representation of Π amounts to the same thing as giving a representation of P which is trivial on elements of the centralizer of G in P. We shall therefore suppose that W is a representation of P. The consistency requirement is that, restricted to P_α, Δ^α should contain W. If this is satisfied, then to get the appropriate representation of H, we have to take a direct integral of induced representations running over one double coset in each Π orbit.

As an example of this we take G to be the Poincaré group $\boldsymbol{R}^4ⓢSL(2, \boldsymbol{C})$, K the little group $\boldsymbol{R}^4ⓢSU(2)$, H the Euclidean subgroup $\boldsymbol{R}^3ⓢSU(2)$, and P the subgroup of time translations in $\boldsymbol{R}^4$. There is no difficulty in showing that all our requirements are fulfilled. Indeed since $P \subseteq K$ we may set $\Delta(p)$ equal to the character $D(p)$. Being in a semi-direct product situation we may take our double coset representatives α to be in $SL(2, \boldsymbol{C})$. Then $\alpha p\alpha^{-1}$ lies on the Lorentz transformed time axis and is certainly in $\boldsymbol{R}^4 \subseteq K$. The consistency requirement is then that $W(p) = D(\alpha p\alpha^{-1})$ for all time translations p. This effectively fixes the energy, and there is only one double coset representative which is compatible with it. We therefore have no need of a direct integral to give the representation of the Euclidean group H as there is only one non-vanishing contribution. Moreover, since $\Pi_\alpha = \Pi(\alpha) = \Pi$ that one contribution is just induced from D on $H \cap K^\alpha = \boldsymbol{R}^3ⓢSO(2)$. This is an irreducible representation. In short, specification of the energy of a relativistic elementary particle specifies irreducible Euclidean behaviour. (We have implicitly assumed here that the energy given is above the rest mass of the particle otherwise no α will do.)

Real representations of complex matrix groups can be investigated by taking Π to be the group generated by complex conjugation. H then has to be a real subgroup and K one which is self-conjugate. Two natural generalizations of this case suggest themselves. One can look at matrix groups over a field extension having Galois group Π. More interestingly one can study semi-simple Lie groups in which Π is generated by a Cartan involution.

An example of this is provided by the Lorentz group $SL(2, \mathbb{C})$, on which the involution $A \mapsto A^{*-1}$ fixes $SU(2)$. Let us suppose that we induce a representation from $SU(2)$ and then restrict it back down to $SU(2)$ again but taking only the part which behaves in a particular way under the Cartan involution. (This sort of situation might arise if we worked with the representations of the previous example but restricted our attention from the Poincaré to the Lorentz group.) We may easily check that all the requirements for the application of the general theory are satisfied. (D can be extended trivially because $SU(2)$ is unaffected by the involution.) The double coset representatives for $SU(2)\backslash SL(2, \mathbb{C})/SU(2)$ can be taken to be real diagonal matrices with first entry greater than one. Any such α is equivalent to its α^{*-1} by conjugation by an element of $SU(2)$. Thus $SU(2)\alpha^{*-1}SU(2) = SU(2)\alpha SU(2)$ and the double coset is unchanged by the involution. On the one hand, this means that we are not able to cut down the number of double cosets required but on the other hand it also means that $\Pi(\alpha)$ is always equal to Π, whilst $\Pi_\alpha = \Pi$ only if $\alpha = 1$, so that we can always induce from a larger subgroup that would otherwise be the case. In fact, when one goes through the details, one discovers that the representation space for $SU(2)$ instead of being all L^2 functions on the sphere is just those which have a specified behaviour under the antipodal map.

Perhaps neither of these examples is particularly exciting in itself, but their range does suggest that there may well be useful applications of the general results besides that for which they were developed.

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